63-2-4

SP-1067/001/00

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SYSTEM DEVELOPMENT CORPORATION, SANTA MONICA, CALIFORNI

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ABSTRACT

The existence of periodic orbits near the critical inclination angle for the equations of motion of a satellite of an oblate planet are herein investigated. It is shown that in the immediate neighborhood of the critical angle these periodic orbits are librations. The period of these orbits is shown to be extremely long, i.e., of order $1/J_2^{3/2}$.

Introduction

Many authors have treated the general problem of the motion of an artificial satellite in the neighborhood of the critical inclination angle. See Garfinkel (1960), Hagihara (1961), Hori (1960), Izsak (1962), Kozai (1961), for a background in this interesting problem.

In the present paper we restrict ourselves to the study of those special orbits which are periodic.

The present paper is an extension of the author's previous paper on the existence of periodic orbits away from the critical inclination angle (see Barrar (1963)). The notation and some of the results of that paper will be used in the present paper. For example, our meaning of a periodic orbit will be that of Barrar (1963). (In Barrar (1963), some results were obtained for motion exactly at the critical inclination angle; the present paper extends these results to a neighborhood of the critical inclination angle.)

Statement of Problem

We assume that after a change of variables (removing the short periodic terms) our Hamiltonian can be written in the form (see Barrar (1963)), equation (36):

$$F = C_o(a_1) + \epsilon\, C_1(a_1,a_2) + \epsilon^2[C_{21}(a_1,a_2) + C_{22}(a_1,a_2)\cos 2w_2$$

$$(1) \qquad + C_{23}(a_1,a_2,w_1,w_2)] + \sum_{i=3}^{\infty} \epsilon^i C_i(a_1,a_2,w_1,w_2)$$

where

$$(2) \qquad \begin{aligned} C_{23}(a_1,a_2,w_1,w_2) &= \sum_{m_1 \neq 0} A' \cos\,(m_1 w_1 + m_2 w_2 + h(a_1,a_2)) \\ C_i(a_1,a_2,w_1,w_2) &= \Sigma\, A'' \cos\,(m_1 w_1 + m_2 w_2 + h(a_1,a_2)) \end{aligned}$$

Further we assume (corresponding to the critical angle) that there exists a point a_1^o, a_2^o such that:

$$(3) \qquad n_1^1 = \partial C_1(a_1^o,a_2^o)/\partial a_1 \neq 0 \qquad\qquad n_2^1 = \partial C_1(a_1^o,a_2^o)/\partial a_2 = 0$$

but

$$(4) \qquad \begin{aligned} A &= \partial^2 C_1(a_1^o,a_2^o)/\partial a_2 \partial a_2 \neq 0 \qquad\qquad D = \partial^2 C_1(a_1^o,a_2^o) \partial a_1 \partial a_1 \neq 0 \\ B &= \partial^2 C_1(a_1^o,a_2^o)/\partial a_1 \partial a_2 = 0 \end{aligned}$$

Since n_2^1 vanishes, the method developed by Poincaré (1893 §134) (i.e., the von-Zeipel method) for non-critical angles no longer applies. However, Poincaré (1893 §206) develops another method that is applicable to the Hamiltonian (1) together with (3) and (4). This method converts the Hamiltonian (1) into another Hamiltonian whose form is the same as that of the original Hamiltonian but with the corresponding n_1^1 and n_2^1 not vanishing. Poincaré calls this procedure the method of Bohlin. We give a presentation of this method below, applicable to the Hamiltonian (1), (2), (3) and (4).

The Poincaré-Bohlin Transformation

As motivation for the Poincaré-Bohlin transformation let us first consider the Hamiltonian:

$$F^*(a_1,a_2,w_1,w_2) = C_1(a_1,a_2) + \epsilon[C_{21}(a_1,a_2) + C_{22}(a_1,a_2)\cos 2w_2 + C_{23}(a_1,a_2,w_1,w_2)] \tag{5}$$

together with the conditions (3) and (4). We introduce $S = S_o + \epsilon^{1/2} S_1 + \epsilon S_2$ with $S_o = a_1^o w_1 + a_2^o w_2$ and wish to solve the equation:

$$F^*(\partial S/\partial w_1, \partial S/\partial w_2, w_1, w_2) = D_o + \epsilon^{1/2} D_1 + \epsilon D_2 + 0(\epsilon^{3/2}) \tag{6}$$

If like powers of $\epsilon^{1/2}$ are equated in this equation, one obtains the series of equations:

$$C_1(a_1^o, a_2^o) = D_o(a_1^o, a_2^o) \tag{7}$$

$$n_1^1\, \partial S_1/\partial w_1 = D_1 \tag{8}$$

$$n_1^1 \partial S_2/\partial w_1 + (A(\partial S_1/\partial w_2)^2 + D(\partial S_1/\partial w_1)^2) = - C_{21}(a_1^o,a_2^o) - C_{22}(a_1^o,a_2^o) \cos 2w_2$$
$$+ D_2 - C_{23}(a_1^o,a_2^o,w_1,w_2) . \tag{9}$$

It follows from this last equation that in order for S_2 not to have any secular terms one must have:

$$A(\partial S_1/\partial w_2)^2 + D(\partial S_1/\partial w_1)^2 = D_2 - C_{21}(a_1^o,a_2^o) - C_{22}(a_1^o,a_2^o) \cos 2w_2 \tag{10}$$

Now if we solve only (7), (8) and (10) and set $S^* = S_o + \epsilon^{1/2} S_1$, we obtain

$$F^*(\partial S^*/\partial w_1,\ \partial S^*/\partial w_2,\ w_1,w_2) = D_o(a_1^o,a_2^o) + \epsilon^{1/2} D_1$$
$$+ \epsilon(D_2 - C_{23}(a_1^o,a_2^o,w_1,w_2)) + O(\epsilon^{3/2}) \tag{11}$$

In the above a_1^o, a_2^o are constants. If in terms of new variables x_1', x_2', we can make $D_1 = D_1(x_1')$, $D_2 = D_2(x_2', x_1')$ with $\partial D_2/\partial x_2' \neq 0$, then in terms of these new variables we are no longer faced with the critical inclination angle difficulties. We do this more precisely below. (The above is a slight modification of the method used in Poincaré (1893) §206. Since Poincaré treats a Hamiltonian of the form (5), but we must treat one of the form (1), this modification is essential.)

We now transfer from the variables a_1,a_2,w_1,w_2 to new variables x_1', x_2', y_1', y_2' by means of a generating function

$$T(a_1^o,a_2^o,x_1',x_2',w_1,w_2) = a_1^o w_1 + a_2^o w_2 + \epsilon^{1/2} T_1 \tag{12}$$

with T_1 satisfying:

(13) $$\partial T_1/\partial w_1 = x_1'$$

(14a) $$A(\partial T_1/\partial w_2)^2 + D(x_1')^2 = D_2 - C_{21}(a_1^o,a_2^o) - C_{22}(a_1^o,a_2^o) \cos 2w_2$$

Further set:

(14b) $$C_{21}(a_1^o,a_2^o) + C_{22}(a_1^o,a_2^o) \cos 2w_2 = A\,\psi$$

(14c) $$x_2' = (D_2 - D(x_1')^2)/A$$

Then,

(15) $$T_1 = x_1'w_1 + \int dw_2\ (x_2'-\psi)^{1/2}$$

[Thus, in (11) $D_1 = n_1x_1'$; $D_2 = A(x_2' + D(x_1')^2)$.]

The canonical variables x_i', y_i' are determined from the equations:

(16) $$a_i = \partial T/\partial w_i \qquad y_i' = \partial T/\partial x_i'$$

or

(17a) $$a_1 = a_1^o + \epsilon^{1/2}\ x_1'$$

(17b) $$y_1' = \epsilon^{1/2}\ w_1$$

(17c) $$a_2 = a_2^o + \epsilon^{1/2}(x_2'-\psi)^{1/2}$$

(17d) $$y_2' = (\epsilon^{1/2}/2)\int (x_2'-\psi)^{-1/2}dw_2$$

[For the physical meaning of this transformation see Poincaré (1893) §199, §200, and Hagihara (1961), discussion of Figure 1.]

Because of the form of ψ (see (14b)), integral (17d) is an elliptic integral, and sin w_2, cos w_2 are doubly periodic in y_2'. If x_2' is greater than the maximum of ψ, the real period is

$$\epsilon^{1/2}P(x_2') = (\epsilon^{1/2}/2)\int_0^{2\Pi} (x_2'-\psi)^{-1/2}dw_2 \tag{18a}$$

However, if $x_2'-\psi$ vanishes for $w_2 = \alpha$, $w_2 = \beta$ and remains positive for $\alpha < w_2 < \beta$, then the real period is

$$\epsilon^{1/2}P(x_2') = (\epsilon^{1/2}/2)\int_\alpha^\beta (x_2'-\psi)^{-1/2}dw_2 \tag{18b}$$

In the case (18a) w_2 increases by 2Π when y_2' increases by a period. In the case (18b), that is in the case of libration, w_2 returns to its primitive value when y_2' increases by a period. Hence, the variables x_1',x_2',y_1',y_2' are uniformizing variables for the original variables a_1,a_2,w_1,w_2.

Several interesting examples arise in the motion of an artificial satellite near the critical inclination angle as noted by Hori (1960). First, when only the second harmonic is considered then only (18a) applies. Secondly, when the Vinti (1959) potential is considered the terms involving cosines and sines of w_2 in ψ vanish and hence (17d) is essentially a linear transformation, with (18a) applying. Finally when both the second and fourth harmonics of the earth's potential are considered, then for sufficiently small values of x_2', the case of libration (18b) applies.

If one makes the change of variable (17) in the Hamiltonian (1), and takes (11) into account, he finds:

$$F = C_o(a_1) + \epsilon\, C_1(a_1,a_2) + \epsilon^2[C_{21}(a_1,a_2) + C_{22}(a_1,a_2)\cos 2w_2 + C_{23}(a_1,a_2,w_1,w_2)] + O(\epsilon^3) \tag{19}$$

$$= C_o(a_1^o) + \epsilon^{1/2}F^* \quad (1)$$

with

$$F^* = \sum_{j=1}^{4}\epsilon^{(j-1)/2}C_o^{(j)}(a_1^o)(x_1')^j/j! + \epsilon\, n_1^1 x_1' + \epsilon^{3/2}A(x_2' + D(x_1')^2) + \epsilon^{3/2}C_{23}(a_1^o,a_2^o,w_1,w_2) + O(\epsilon^2) \tag{20}$$

where

$$C^{(j)}(a_1^o) \equiv d^jC_o(a_1^o)/(da_1)^j$$

Further if one introduces the change of variable

$$y_i' = z_i\epsilon^{1/2} \tag{21}$$

and since $C_o(a_1^o)$ in (19) is a constant the equations of motion become:

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(1) The present method of dealing first with (5) and then with (1), in making a change of variable, was used to avoid the difficulty mentioned in the introduction of Izsak (1962) with terms of order $\epsilon^{3/2}$.

(22) $$dx_i'/dt = \partial F^*/\partial z_i \qquad dz_i/dt = -\partial F^*/\partial x_i'$$

Although F^* is periodic with respect to z_1, z_2, the period with respect to z_1 is 2Π, but with respect to z_2 it is $P(x_2')$ as defined in (18). To make the period 2Π with respect to both angle variables we introduce the last canonical variable change:

(23) $$u_2 = \int P(x_2')dx_2'/2\Pi \quad \nu_2 = 2\Pi z_2/P(x_2')$$

The equations remain canonical:

(24) $$\begin{aligned} dx_1'/dt &= \partial F^*/\partial z_1 & du_2/dt &= \partial F^*/\partial \nu_2 \\ dz_1/dt &= -\partial F^*/\partial x_1' & d\nu_2/dt &= -\partial F^*/\partial u_2 . \end{aligned}$$

F^* is periodic with period 2Π with respect to z_1, ν_2, and the original variables a_1, a_2, w_1, w_2 can be expressed in terms of x_1', u_2, z_1, ν_1 and are periodic with period 2Π with respect to z_1, ν_2.

Periodic Solutions

By means of the Poincaré-Bohlin transformation we have changed the Hamiltonian (1) to one of the form:

(25) $$F^* = E_o(x_1') + \epsilon^{1/2}E_1(x_1') + \epsilon E_2(x_1') + \epsilon^{3/2}(E_{31}(x_1', u_2) + E_{32}(x_1', u_2, z_1, \nu_2)) + \sum_{j=4}^{\infty} \epsilon^{j/2} E_j(x_1', u_2, z_1, \nu_2)$$

where

$$E_{32} = \sum_{m_1 \neq 0} A'(x_1', u_2) \cos (m_1 z_1 + m_2 \nu_2 + h(x_1', u_2))$$

$$E_j = \Sigma A''(x_1', u_2) \cos (m_1 z_1 + m_2 \nu_2 + h(x_1', u_2))$$

with

$$d E_o(x_1')/dx_1' \neq 0 \qquad \partial E_{31}(x_1', u_2)/\partial u_2 \neq 0 .$$

This Hamiltonian (25) is slightly different than the Hamiltonian treated by means of the Poincaré--von-Zeipel transformation, in Barrar (1963), in that the variable u_2 does not occur until the fourth term. Nevertheless the Poincaré--von-Zeipel transformation is applicable to the Hamiltonian (25) with some obvious modifications such as expanding in powers of $\epsilon^{1/2}$ not ϵ. Thus from the results of Barrar (1963), we conclude that (25) has periodic solutions for arbitrary initial values of x_1', u_2, z_1 and for values of ν_2 satisfying the Poincaré criteria (19b) of Barrar (1963). Since the original variables a_1, a_2, w_1, w_2 are periodic in x_1', u_2, z_1, ν_2 this also insures periodic solutions in terms of the original variables.

Note that the period of the resulting motion will be very long since the frequency with respect to ν_2 is $O(\epsilon^{3/2})$ of the frequency with respect to z_1.

Conclusion

We have studied periodic orbits in the immediate neighborhood of the critical inclination angle for the motion of a satellite of an oblate planet. By means of the Poincaré-Bohlin transformation we have been able to introduce new uniformizing variables, such that the equations of motion in terms of the new variables have essentially the same form as the equations of motion in terms of the old variables but away from the critical angle. From the results of Barrar (1963) we are then able to conclude the existence of periodic orbits near the critical inclination angle. Very near the critical angle these periodic orbits are librations.

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System Development Corporation,
Santa Monica, California
PERIODIC ORBITS NEAR THE CRITICAL
INCLINATION ANGLE.
Scientific rept., SP-1067/001/00,
by R. B. Barrar. 28 January 1963,
11p., 8 refs.

Unclassified report

DESCRIPTORS: Satellites. Mathematical
Analysis.

Investigates the existence of periodic
orbits near the critical inclination

UNCLASSIFIED

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